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ON THE FREE-EDGE STRESS SINGULARITY OF GENERAL COMPOSITE LAMINATES UNDER UNIFORM AXIAL STRAIN

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Abstract—This work presents a rigorous study on the free-edge stress singularity for general composite laminates (except the angle-ply and cross-ply laminates) under uniform axial strain. Based on the general solutions derived for the plies with 0°, 90° and arbitrary fiber orientations, the corresponding transcendental characteristic equations of general $[\theta_1/\theta_2]$, $[\theta/0°]$ and $[\theta/90°]$ laminates are established by using appropriate near-field boundary conditions, respectively. To avoid numerical errors, the transcendental characteristic equations are expanded by a symbolic operation technique analytically and solved graphically. Hence, once the ply material properties and stacking sequence are known, the accurate characteristics of the free-edge stress singularity of examined composite laminates can be obtained. These will be helpful for the failure analysis of composite materials and structures.

1. INTRODUCTION

The stress analysis of composite laminates in the vicinity of free edges is of interest to structural designers. The potentially high stress gradients which exist near these regions may limit the load carrying capability of the structure, and will, in general, be a source of laminate failure. Although many distinguished researchers have been devoted to study the free-edge stress singularity in the literature (Reissner, 1944; Bogy, 1968; Hein and Erdogan, 1971; Kuo and Bogy, 1974; Yin, 1991), only some simpler specific laminates (for example, angle-ply and cross-ply laminates) are tackled. For other general composite laminates, the free-edge stress singularity is therefore worthy of further exploration.

Using Lekhnitskii's stress potentials (Lekhnitskii, 1963), Wang and Choi (1982) analysed the singularity order of the free-edge stresses for a composite laminate under uniform axial strain. To obtain the singularity order, the numerical Muller's method (Muller, 1956) was used. However, due to the drawbacks of such a numerical technique (Muller, 1956) in root convergence, the accuracy of the singularity order obtained is doubtful. Zwiers *et al.* (1982) employed Stroh's formalism (Stroh, 1962) to study the same problem. In addition to the same free-edge stress singularity as obtained by Wang and Choi (1982), they further indicated that a logarithmic singularity may exist when the roots of the transcendental characteristic equation contain a double root. Since the numerical Muller's method was again adopted, the same doubtful singularity order as obtained by Wang and Choi (1982) was also noted. More recently, to compensate the deficiency, Chen and Huang (1993a, b) successfully used an analytical approach on the free-edge stress singularity and the free-edge effects of the stress distribution for the angle-ply and cross-ply laminates under uniform axial strain. Unfortunately, due to the different characteristics of fiber orientations, such an approach can not be implemented for other laminates directly or easily. The free-edge stress singularity for other commonly used composite laminates was hard to obtain because of its high complexity and very little attention has been paid in the literature. The aim of this work is thus to present a rigorous investigation on the free-edge stress singularity for general composite laminates except angle-ply and cross-ply laminates.

Based on the general solutions derived for the plies with 0°, 90° and arbitrary fiber orientations under uniform axial strain (Chen and Huang, 1993a, b), the free-edge stress singularity of general $[\theta_1/\theta_2]$, $[\theta/0°]$ and $[\theta/90°]$ laminates is studied by using appropriate

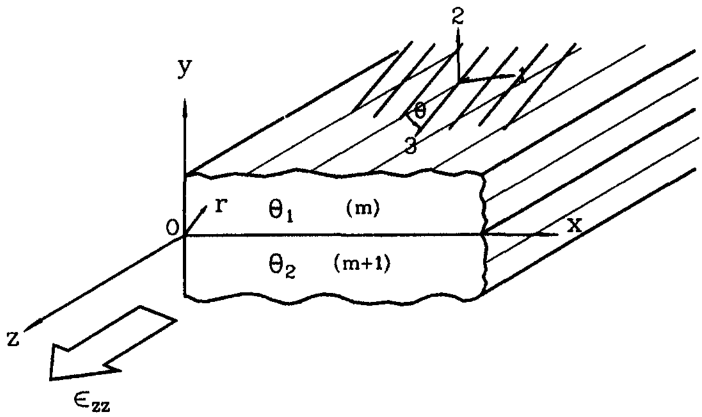

Fig. 1. The composite laminate under uniform axial strain.

near-field boundary conditions. The transcendental characteristic equation $|\mathbf{K}(\delta)| = 0$ is established for each type of laminate, respectively. $\mathbf{K}(\delta)$ is a 12×12 matrix of the complex constant δ for the general $[\theta_1/\theta_2]$ laminate. For the cases of $[\theta/0^\circ]$ and $[\theta/90^\circ]$ laminates, however, $\mathbf{K}(\delta)$ is a 10×10 matrix. The order and types of singularity depend on the magnitudes of the real and imaginary parts of the complex constant δ. To avoid numerical errors in determining the complex constant δ, those transcendental characteristic equations are expanded by a symbolic operation technique (Pavelle and Wang, 1985) analytically instead of the numerical Muller's method (Wang and Choi, 1982; Zwiers *et al.*, 1982). After solving the transcendental characteristic equations graphically, once the ply material properties are known, the free-edge stress singularity for any examined composite laminates under uniform axial strain can be determined.

To comprehend the singular nature of the free-edge stress field, the commonly used graphite–epoxy laminates under uniform axial strain are examined as an example. The free-edge stress singularity obtained leads to the near-field solution which may assist the designer in fully understanding the local stress distribution near the free edge.

2. GENERAL SOLUTIONS OF INDIVIDUAL PLY

2.1. *The ply with orientations other than* 0° *and* 90°

As seen in Fig. 1, the composite laminate considered is subjected to a uniform extensional strain $\varepsilon_{zz} = e$. Since the composite is sufficiently long, the end effect is negligible. The displacements (except the displacement in the z direction, w) and strains and stresses in the laminate are hence independent of the z-coordinate. Moreover, each ply of the composite laminate is made of orthotropic materials. Based on the merits of Lekhnitskii's approach, the stress field near the free edge is assumed to be proportional to a r^δ form. Lekhnitskii's stress functions are reconsidered in this work by expressing the complex constant δ in terms of its real and imaginary parts as $\delta = \delta_r + i\delta_i$. Under these assumptions, for completeness, the general solution of stresses and displacements for the ply with arbitrary orientations other than 0° and 90° derived by Chen and Huang (1993b) is quoted as follows:

$$\sigma_{xx} = \sum_{k=1}^{3} \{C_k\mu_k^2 Z_{k\mu}^{\delta_r}[\cos(\delta_i \ln Z_{k\mu}) + i\sin(\delta_i \ln Z_{k\mu})] + \bar{C}_k\bar{\mu}_k^2\bar{Z}_{k\mu}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i\sin(\delta_i \ln \bar{Z}_{k\mu})]\}$$

$$\sigma_{yy} = \sum_{k=1}^{3} \{C_k Z_{k\mu}^{\delta_r}[\cos(\delta_i \ln Z_{k\mu}) + i\sin(\delta_i \ln Z_{k\mu})] + \bar{C}_k\bar{Z}_{k\mu}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i\sin(\delta_i \ln \bar{Z}_{k\mu})]\}$$

$$\tau_{yz} = -\sum_{k=1}^{3} \{C_k \eta_k Z_{k\mu}^{\delta_r}[\cos(\delta_i \ln Z_{k\mu}) + i \sin(\delta_i \ln Z_{k\mu})] + \bar{C}_k \bar{\eta}_k \bar{Z}_{k\mu}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i \sin(\delta_i \ln \bar{Z}_{k\mu})]\}$$

$$\tau_{xz} = \sum_{k=1}^{3} \{C_k \eta_k \mu_k Z_{k\mu}^{\delta_r}[\cos(\delta_i \ln Z_{k\mu}) + i \sin(\delta_i \ln Z_{k\mu})] + \bar{C}_k \bar{\eta}_k \bar{\mu}_k \bar{Z}_{k\mu}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i \sin(\delta_i \ln \bar{Z}_{k\mu})]\}$$

$$\tau_{xy} = -\sum_{k=1}^{3} \{C_k \mu_k Z_{k\mu}^{\delta_r}[\cos(\delta_i \ln Z_{k\mu}) + i \sin(\delta_i \ln Z_{k\mu})] + \bar{C}_k \bar{\mu}_k \bar{Z}_{k\mu}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i \sin(\delta_i \ln \bar{Z}_{k\mu})]\}$$

$$\sigma_{zz} = (e - S_{13}\sigma_{xx} - S_{23}\sigma_{yy} - S_{35}\tau_{xz})\frac{1}{S_{33}} \tag{1}$$

and

$$u = \sum_{k=1}^{3} \left\{ \frac{C_k p_{k\mu} Z_{k\mu}^{\delta_r+1}}{[(\delta_r+1)+i\delta_i]}[\cos(\delta_i \ln Z_{k\mu}) + i \sin(\delta_i \ln Z_{k\mu})] + \frac{\bar{C}_k \bar{p}_{k\mu} \bar{Z}_{k\mu}^{\delta_r+1}}{[(\delta_r+1)-i\delta_i]}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i \sin(\delta_i \ln \bar{Z}_{k\mu})] \right\} + \frac{S_{13}}{S_{33}} ex - \omega_3 y + u_0$$

$$v = \sum_{k=1}^{3} \left\{ \frac{C_k q_{k\mu} Z_{k\mu}^{\delta_r+1}}{[(\delta_r+1)+i\delta_i]}[\cos(\delta_i \ln Z_{k\mu}) + i \sin(\delta_i \ln Z_{k\mu})] + \frac{\bar{C}_k \bar{q}_{k\mu} \bar{Z}_{k\mu}^{\delta_r+1}}{[(\delta_r+1)-i\delta_i]}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i \sin(\delta_i \ln \bar{Z}_{k\mu})] \right\} + \omega_3 x + \frac{S_{23}}{S_{33}} ey + v_0$$

$$w = \sum_{k=1}^{3} \left\{ \frac{C_k t_{k\mu} Z_{k\mu}^{\delta_r+1}}{[(\delta_r+1)+i\delta_i]}[\cos(\delta_i \ln Z_{k\mu}) + i \sin(\delta_i \ln Z_{k\mu})] + \frac{\bar{C}_k \bar{t}_{k\mu} \bar{Z}_{k\mu}^{\delta_r+1}}{[(\delta_r+1)-i\delta_i]}[\cos(\delta_i \ln \bar{Z}_{k\mu}) - i \sin(\delta_i \ln \bar{Z}_{k\mu})] \right\} + \left(\frac{S_{35}}{S_{33}} e - \omega_2\right)x + \omega_1 y + ez + w_0, \tag{2}$$

where the overbar denotes the complex conjugate of the associated quantity and the complex variable $Z_{k\mu} = x + \mu_k y$ which can be considered as a normalized dimensionless variable for dimensional consideration. The constants (u_0, v_0, w_0) and $(\omega_1, \omega_2, \omega_3)$ denote the rigid-body translation and rotation of the ply, respectively. The complex coefficients μ_k and η_k are the roots of the following algebraic characteristic equations:

$$l_4(\mu)l_2(\mu) - l_3^2(\mu) = 0$$

and

$$\eta_k = -\frac{l_3(\mu_k)}{l_2(\mu_k)} = -\frac{l_4(\mu_k)}{l_3(\mu_k)},$$

where

$$l_2(\mu) = \beta_{55}\mu^2 + \beta_{44}$$

$$l_3(\mu) = \beta_{15}\mu^3 + (\beta_{25} + \beta_{46})\mu$$

and

$$l_4(\mu) = \beta_{11}\mu^4 + (2\beta_{12} + \beta_{66})\mu^2 + \beta_{22}.$$

In addition, the complex coefficients $p_{k\mu}$, $q_{k\mu}$ and $t_{k\mu}$ are defined as

$$p_{k\mu} = \beta_{11}\mu_k^2 + \beta_{12} + \beta_{15}\eta_k\mu_k$$

$$q_{k\mu} = \beta_{12}\mu_k + \beta_{22}/\mu_k + \beta_{25}\eta_k$$

and

$$t_{k\mu} = -\beta_{44}\eta_k/\mu_k - \beta_{46},$$

where $\beta_{ij} = S_{ij} - (S_{i3}S_{j3}/S_{33})$ $(i, j = 1\text{–}6)$ is the anisotropic elastic constant of each individual ply. The compliance matrix of an anisotropic material S_{ij} can be written as $\varepsilon_i = S_{ij}\sigma_j$, where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}$ and $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{yz}, \tau_{xz}, \tau_{xy}\}$. The stresses and displacements expressed in eqns (1) and (2) can be proved as real after necessary manipulations. Consequently, once the unknown coefficients C_k, $\bar{C}_k$, δ_r and δ_i are determined by given boundary conditions of the corresponding composite laminate, the complete solutions of the stresses and displacements throughout the entire domain of the composite laminate tackled can be determined.

2.2. *The* 0° *and* 90° *plies*

Since the general solution as mentioned above is not valid for the 0° and 90° plies, the general solutions of those two plies need to be resolved and presented as (Chen and Huang, 1993a)

$$\sigma_{xx} = \sum_{k=1}^{2} \{C_k\zeta_k^2 Z_{k\zeta}^{\delta_r}[\cos(\delta_i \ln Z_{k\zeta}) + i\sin(\delta_i \ln Z_{k\zeta})]$$
$$+ \bar{C}_k\bar{\zeta}_k^2 \bar{Z}_{k\zeta}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\zeta}) - i\sin(\delta_i \ln \bar{Z}_{k\zeta})]\}$$

$$\sigma_{yy} = \sum_{k=1}^{2} \{C_k Z_{k\zeta}^{\delta_r}[\cos(\delta_i \ln Z_{k\zeta}) + i\sin(\delta_i \ln Z_{k\zeta})]$$
$$+ \bar{C}_k \bar{Z}_{k\zeta}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\zeta}) - i\sin(\delta_i \ln \bar{Z}_{k\zeta})]\}$$

$$\tau_{yz} = -\{C_3 Z_{1\xi}^{\gamma_r}[\cos(\gamma_i \ln Z_{1\xi}) + i\sin(\gamma_i \ln Z_{1\xi})]$$
$$+ \bar{C}_3 \bar{Z}_{1\xi}^{\gamma_r}[\cos(\gamma_i \ln \bar{Z}_{1\xi}) - i\sin(\gamma_i \ln \bar{Z}_{1\xi})]\}$$

$$\tau_{xz} = \{C_3\xi_1 Z_{1\xi}^{\gamma_r}[\cos(\gamma_i \ln Z_{1\xi}) + i\sin(\gamma_i \ln Z_{1\xi})]$$
$$+ \bar{C}_3\bar{\xi}_1 \bar{Z}_{1\xi}^{\gamma_r}[\cos(\gamma_i \ln \bar{Z}_{1\xi}) - i\sin(\gamma_i \ln \bar{Z}_{1\xi})]\}$$

$$\tau_{xy} = \sum_{k=1}^{2} \{C_k\zeta_k Z_{k\zeta}^{\delta_r}[\cos(\delta_i \ln Z_{k\zeta}) + i\sin(\delta_i \ln Z_{k\zeta})]$$
$$+ \bar{C}_k\bar{\zeta}_k \bar{Z}_{k\zeta}^{\delta_r}[\cos(\delta_i \ln \bar{Z}_{k\zeta}) - i\sin(\delta_i \ln \bar{Z}_{k\zeta})]\}$$

$$\sigma_{zz} = (e - S_{13}\sigma_{xx} - S_{23}\sigma_{yy})\frac{1}{S_{33}} \tag{3}$$

and

$$u = \sum_{k=1}^{2}\left\{\frac{C_k p_{k\zeta} Z_{k\zeta}^{\delta_r+1}}{[(\delta_r+1)+i\delta_i]}[\cos(\delta_i \ln Z_{k\zeta})+i\sin(\delta_i \ln Z_{k\zeta})]\right.$$

$$\left.+\frac{\bar{C}_k \bar{p}_{k\zeta} \bar{Z}_{k\zeta}^{\delta_r+1}}{[(\delta_r+1)-i\delta_i]}[\cos(\delta_i \ln \bar{Z}_{k\zeta})-i\sin(\delta_i \ln \bar{Z}_{k\zeta})]\right\}+\frac{S_{13}}{S_{33}}ex-\omega_3 y+u_0$$

$$v = \sum_{k=1}^{2}\left\{\frac{C_k q_{k\zeta} Z_{k\zeta}^{\delta_r+1}}{[(\delta_r+1)+i\delta_i]}[\cos(\delta_i \ln Z_{k\zeta})+i\sin(\delta_i \ln Z_{k\zeta})]\right.$$

$$\left.+\frac{\bar{C}_k \bar{q}_{k\zeta} \bar{Z}_{k\zeta}^{\delta_r+1}}{[(\delta_r+1)-i\delta_i]}[\cos(\delta_i \ln \bar{Z}_{k\zeta})-i\sin(\delta_i \ln \bar{Z}_{k\zeta})]\right\}+\omega_3 x+\frac{S_{23}}{S_{33}}ey+v_0$$

$$w = \left\{\frac{C_3 t_{1\xi} Z_{1\xi}^{\gamma_r+1}}{[(\gamma_r+1)+i\gamma_i]}[\cos(\gamma_i \ln Z_{1\xi})+i\sin(\gamma_i \ln Z_{1\xi})]\right.$$

$$\left.+\frac{\bar{C}_3 \bar{t}_{1\xi} \bar{Z}_{1\xi}^{\gamma_r+1}}{[(\gamma_r+1)-i\gamma_i]}[\cos(\gamma_i \ln \bar{Z}_{1\xi})-i\sin(\gamma_i \ln \bar{Z}_{1\xi})]\right\}-\omega_2 x+\omega_1 y+ez+w_0, \quad (4)$$

where the complex coefficients ζ_k and ξ_k of the complex variables $Z_{k\zeta} = x+\zeta_k y$ and $Z_{k\xi} = x+\xi_k y$ are the roots of the following algebraic characteristic equations:

$$\beta_{11}\zeta^4+(2\beta_{12}+\beta_{66})\zeta^2+\beta_{22} = 0$$

$$\beta_{55}\xi^2+\beta_{44} = 0.$$

Thus, the complex coefficients $p_{k\zeta}$, $q_{k\zeta}$ and $t_{k\xi}$ are given as

$$p_{k\zeta} = \beta_{11}\zeta_k^2+\beta_{12}$$

$$q_{k\zeta} = \beta_{12}\zeta_k+\beta_{22}/\zeta_k$$

and

$$t_{k\xi} = \beta_{55}\xi_k.$$

The stresses and displacements as expressed in eqns (3) and (4) can be determined when the unknown coefficients C_k, $\bar{C}_k$, δ_r, δ_i, γ_r and γ_i are determined by given boundary conditions.

3. STRESS SINGULARITY

Due to the composite laminate bonded by two different fiber orientations, the stress singularity inherent to the material and geometric discontinuities of general composite laminates should be evaluated specifically. The complex constants $\delta = \delta_r+i\delta_i$ and $\gamma = \gamma_r+i\gamma_i$ which appeared in the general solutions of eqns (1)–(4) can be determined by satisfying the near-field boundary conditions of the corresponding composite laminate. This leads to solving an eigenvalue problem. The near-field boundary conditions of general composite laminates include the traction-free conditions at the free edge and the continuity conditions along the ply interface. The traction-free conditions at the free edge of the (m)th and $(m+1)$th plies are (see Fig. 1)

$$\sigma_{xx}^{(m)} = \tau_{xy}^{(m)} = \tau_{xz}^{(m)} = \sigma_{xx}^{(m+1)} = \tau_{xy}^{(m+1)} = \tau_{xz}^{(m+1)} = 0 \quad \text{on} \quad x = 0, \quad (5)$$

where the superscripts (m) and $(m+1)$ denote the (m)th and $(m+1)$th plies of the composite laminate, respectively. The continuity conditions along the ply interface give

$$\{\sigma_{yy}^{(m)}, \tau_{yz}^{(m)}, \tau_{xy}^{(m)}, u^{(m)}, v^{(m)}, w^{(m)}\} = \{\sigma_{yy}^{(m+1)}, \tau_{yz}^{(m+1)}, \tau_{xy}^{(m+1)}, u^{(m+1)}, v^{(m+1)}, w^{(m+1)}\} \quad \text{on} \quad y = 0. \tag{6}$$

Since the general solutions of the 0° and 90° plies can not be induced from the arbitrary orientation plies, the present study of the free-edge stress singularity for the general composite laminates is divided into two categories of laminates, say $[\theta_1/\theta_2]$ and ($[\theta/0°]$ and $[\theta/90°]$) laminates.

3.1. *General* $[\theta_1/\theta_2]$ *laminates*

From the general solutions of eqns (1) and (2) and the near-field boundary conditions of eqns (5) and (6), a system of the algebraic equation for the general $[\theta_1/\theta_2]$ laminates is obtained as

$$\mathbf{K}(\delta)\mathbf{C} = 0, \tag{7}$$

where $\mathbf{K}(\delta)$ is a 12×12 matrix of the complex constant δ and $\mathbf{C}$ is a 12×1 column vector made of the complex constants $C_k^{(m)}$ and $C_k^{(m+1)}$ ($k = 1$–6). The existing condition for a nontrivial $\mathbf{C}$ is

$$|\mathbf{K}(\delta)| = \begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\mu_{1(m)} & \bar{\mu}_{1(m)} & \mu_{2(m)} & \bar{\mu}_{2(m)} & \mu_{3(m)} & \bar{\mu}_{3(m)} & \mu_{1(m+1)} & \bar{\mu}_{1(m+1)} & \mu_{2(m+1)} & \bar{\mu}_{2(m+1)} & \mu_{3(m+1)} & \bar{\mu}_{3(m+1)} \\
\eta_{1(m)} & \bar{\eta}_{1(m)} & \eta_{2(m)} & \bar{\eta}_{2(m)} & \eta_{3(m)} & \bar{\eta}_{3(m)} & \eta_{1(m+1)} & \bar{\eta}_{1(m+1)} & \eta_{2(m+1)} & \bar{\eta}_{2(m+1)} & \eta_{3(m+1)} & \bar{\eta}_{3(m+1)} \\
p_{1\mu}^{(m)} & \bar{p}_{1\mu}^{(m)} & p_{2\mu}^{(m)} & \bar{p}_{2\mu}^{(m)} & p_{3\mu}^{(m)} & \bar{p}_{3\mu}^{(m)} & p_{1\mu}^{(m+1)} & \bar{p}_{1\mu}^{(m+1)} & p_{2\mu}^{(m+1)} & \bar{p}_{2\mu}^{(m+1)} & p_{3\mu}^{(m+1)} & \bar{p}_{3\mu}^{(m+1)} \\
q_{1\mu}^{(m)} & \bar{q}_{1\mu}^{(m)} & q_{2\mu}^{(m)} & \bar{q}_{2\mu}^{(m)} & q_{3\mu}^{(m)} & \bar{q}_{3\mu}^{(m)} & q_{1\mu}^{(m+1)} & \bar{q}_{1\mu}^{(m+1)} & q_{2\mu}^{(m+1)} & \bar{q}_{2\mu}^{(m+1)} & q_{3\mu}^{(m+1)} & \bar{q}_{3\mu}^{(m+1)} \\
t_{1\mu}^{(m)} & \bar{t}_{1\mu}^{(m)} & t_{2\mu}^{(m)} & \bar{t}_{2\mu}^{(m)} & t_{3\mu}^{(m)} & \bar{t}_{3\mu}^{(m)} & t_{1\mu}^{(m+1)} & \bar{t}_{1\mu}^{(m+1)} & t_{2\mu}^{(m+1)} & \bar{t}_{2\mu}^{(m+1)} & t_{3\mu}^{(m+1)} & \bar{t}_{3\mu}^{(m+1)} \\
A_{11}^{(m)} & \bar{A}_{11}^{(m)} & A_{12}^{(m)} & \bar{A}_{12}^{(m)} & A_{13}^{(m)} & \bar{A}_{13}^{(m)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{11}^{(m+1)} & \bar{A}_{11}^{(m+1)} & A_{12}^{(m+1)} & \bar{A}_{12}^{(m+1)} & A_{13}^{(m+1)} & \bar{A}_{13}^{(m+1)} \\
A_{21}^{(m)} & \bar{A}_{21}^{(m)} & A_{22}^{(m)} & \bar{A}_{22}^{(m)} & A_{23}^{(m)} & \bar{A}_{23}^{(m)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{21}^{(m+1)} & \bar{A}_{21}^{(m+1)} & A_{22}^{(m+1)} & \bar{A}_{22}^{(m+1)} & A_{23}^{(m+1)} & \bar{A}_{23}^{(m+1)} \\
A_{31}^{(m)} & \bar{A}_{31}^{(m)} & A_{32}^{(m)} & \bar{A}_{32}^{(m)} & A_{33}^{(m)} & \bar{A}_{33}^{(m)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{31}^{(m+1)} & \bar{A}_{31}^{(m+1)} & A_{32}^{(m+1)} & \bar{A}_{32}^{(m+1)} & A_{33}^{(m+1)} & \bar{A}_{33}^{(m+1)}
\end{vmatrix} = 0, \tag{8}$$

where $A_{1k}^{(m)} = \mu_{k(m)}^{\delta+2}$, $A_{2k}^{(m)} = \mu_{k(m)}^{\delta+1}$, $A_{3k}^{(m)} = \eta_{k(m)}\mu_{k(m)}^{\delta+1}$, $A_{1k}^{(m+1)} = \mu_{k(m+1)}^{\delta+2}$, $A_{2k}^{(m+1)} = \mu_{k(m+1)}^{\delta+1}$ and $A_{3k}^{(m+1)} = \eta_{k(m+1)}\mu_{k(m+1)}^{\delta+1}$ ($k = 1, 2, 3$). Obviously, eqn (8) is a transcendental characteristic equation and $\delta = 0$ is one of the roots. To avoid numerical errors in computing δ, the transcendental characteristic eqn (8) is first expanded by the symbolic operation technique (Pavelle and Wang, 1985) in this work instead of using the numerical Muller's method (Wang and Choi, 1982; Zwiers *et al.*, 1982). Since the transcendental characteristic eqn (8) has a very complicated structure, the detailed expression is not given here.

3.2. $[\theta/0°]$ *and* $[\theta/90°]$ *laminates*

Since the free-edge stress singularity of the $[\theta/0°]$ and $[\theta/90°]$ laminates cannot be induced from the general $[\theta_1/\theta_2]$ laminates directly, the transcendental characteristic equation for these two types of laminates needs to be resolved. To satisfy the continuity conditions of the transverse shear stress τ_{yz} and the displacement w along the ply interface of the (m)th and $(m+1)$th plies, the complex constant γ should be equal to the complex constant δ in the entire domain of the $[\theta/0°]$ and $[\theta/90°]$ laminates. Two of the constants $C_k^{(m+1)}$, say $C_5^{(m+1)}$ and $C_6^{(m+1)}$, can be thus expressed in terms of $C_k^{(m)}$ (k = 1–6). Therefore, the corresponding system of algebraic equations for the $[\theta/0°]$ or $[\theta/90°]$ laminates can be simplified as $\mathbf{K}_d(\delta)\mathbf{C}_d = 0$, where $\mathbf{K}_d(\delta)$ is a 10×10 matrix of the complex constant δ and $\mathbf{C}_d$ is a 10×1 column vector composed of the complex constants $C_k^{(m)}$ (k = 1–6) and $C_j^{(m+1)}$ (j = 1–4). The existing condition for a nontrivial $\mathbf{C}_d$ is thereby

$$|\mathbf{K}_d(\delta)| = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \mu_{1(m)} & \bar{\mu}_{1(m)} & \mu_{2(m)} & \bar{\mu}_{2(m)} & \mu_{3(m)} & \bar{\mu}_{3(m)} & \zeta_{1(m+1)} & \bar{\zeta}_{1(m+1)} & \zeta_{2(m+1)} & \bar{\zeta}_{2(m+1)} \\ p_{1\mu}^{(m)} & \bar{p}_{1\mu}^{(m)} & p_{2\mu}^{(m)} & \bar{p}_{2\mu}^{(m)} & p_{3\mu}^{(m)} & \bar{p}_{3\mu}^{(m)} & p_{1\zeta}^{(m+1)} & \bar{p}_{1\zeta}^{(m+1)} & p_{2\zeta}^{(m+1)} & \bar{p}_{2\zeta}^{(m+1)} \\ q_{1\mu}^{(m)} & \bar{q}_{1\mu}^{(m)} & q_{2\mu}^{(m)} & \bar{q}_{2\mu}^{(m)} & q_{3\mu}^{(m)} & \bar{q}_{3\mu}^{(m)} & q_{1\zeta}^{(m+1)} & \bar{q}_{1\zeta}^{(m+1)} & q_{2\zeta}^{(m+1)} & \bar{q}_{2\zeta}^{(m+1)} \\ A_{11}^{(m)} & \bar{A}_{11}^{(m)} & A_{12}^{(m)} & \bar{A}_{12}^{(m)} & A_{13}^{(m)} & \bar{A}_{13}^{(m)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_{1(m+1)}^{\delta+2} & \bar{\zeta}_{1(m+1)}^{\delta+2} & \zeta_{2(m+1)}^{\delta+2} & \bar{\zeta}_{2(m+1)}^{\delta+2} \\ A_{21}^{(m)} & \bar{A}_{21}^{(m)} & A_{22}^{(m)} & \bar{A}_{22}^{(m)} & A_{23}^{(m)} & \bar{A}_{23}^{(m)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_{1(m+1)}^{\delta+1} & \bar{\zeta}_{1(m+1)}^{\delta+1} & \zeta_{2(m+1)}^{\delta+1} & \bar{\zeta}_{2(m+1)}^{\delta+1} \\ A_{31}^{(m)} & \bar{A}_{31}^{(m)} & A_{32}^{(m)} & \bar{A}_{32}^{(m)} & A_{33}^{(m)} & \bar{A}_{33}^{(m)} & 0 & 0 & 0 & 0 \\ H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

Again, $\delta = 0$ is one of the roots of the transcendental characteristic equation, and the transcendental characteristic equation $|\mathbf{K}_d(\delta)| = 0$ for the $[\theta/0°]$ or $[\theta/90°]$ laminates is expanded as

$$|\mathbf{K}_d(\delta)| = \sum T_{ijklmn} B_j B_k B_l A_m A_n = 0, \tag{9}$$

where the functions T_{ijklmn}, B_j and A_m are given in the Appendix.

The transcendental characteristic eqns (8) and (9) can be solved graphically. Since these transcendental characteristic equations can be sketched as smooth surfaces in the $|\mathbf{K}_d(\delta)| - \delta_r - \delta_i$ space, the eigenvalues δ_n ($n = 1, 2 \ldots$) obtained (on the $|\mathbf{K}_d(\delta)| = 0$ plane) should be a continuous spectrum (see Fig. 2). Hence, once the ply material properties and stacking sequence are known, the accurate characteristics of the free-edge stress singularity of examined composite laminates can be obtained.

3.3. *Dominant free-edge stress singularity*

To understand the singular nature of the free-edge stress field further, the form of trigonometric functions as seen in eqns (1)–(4) can be further represented as

$$\cos(\delta_i \ln Z) = 1 - \frac{\delta_i^2}{2!}(\ln Z)^2 + \frac{\delta_i^4}{4!}(\ln Z)^4 - \frac{\delta_i^6}{6!}(\ln Z)^6 + \frac{\delta_i^8}{8!}(\ln Z)^8 \ldots$$

and

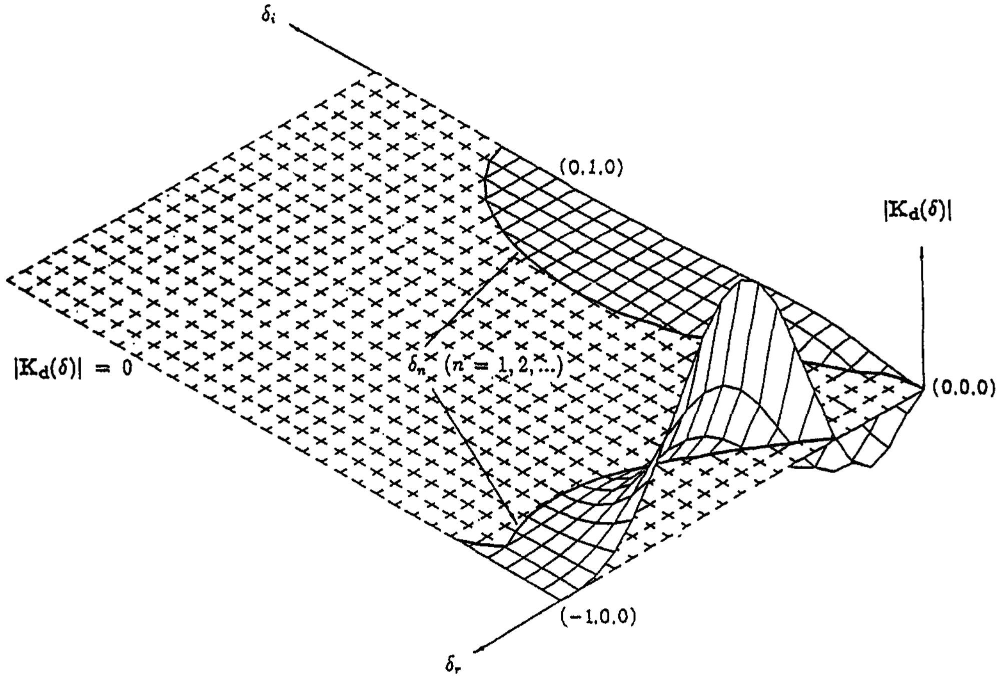

Fig. 2. The graphite solution of the transcendental characteristic equation for [30°/60°] graphite–epoxy laminate.

$$\sin(\delta_i \ln Z) = \delta_i \ln Z - \frac{\delta_i^3}{3!}(\ln Z)^3 + \frac{\delta_i^5}{5!}(\ln Z)^5 - \frac{\delta_i^7}{7!}(\ln Z)^7 \ldots.$$

As seen from eqns (1) and (3), the real parts δ_r and γ_r are nothing but the stress order and the imaginary parts δ_i and γ_i associated with the logarithmic singularity. To prevent the displacement at the point O being infinite (see Fig. 1), the real parts δ_r and γ_r should be > -1. Moreover, the stresses at the point O are found to be singular as $\delta_r < 0$ and $\gamma_r < 0$. Therefore, the conditions $-1 < \delta_r < 0$ and $-1 < \gamma_r < 0$ will characterize the order of stress singularity at the corner of the free-edge interface and the logarithmic singularity appears when the imaginary parts δ_i and γ_i exist.

For the [θ/0°] and [θ/90°] laminates, as stated earlier, the complex constant γ is equal to the complex constant δ to satisfy the continuity conditions of the transverse shear stress τ_{yz} and the displacement w along the ply interface. Thus, the free-edge stress singularity of all the [θ_1/θ_2], [θ/0°] and [θ/90°] laminates may include three elementary forms, say r^{δ_r}, $(\ln r)^n$ and $r^{\delta_r}(\ln r)^n$ ($n = 1, 2\ldots$) depending on the magnitudes of δ_r and δ_i. The occurrence of those three elementary stress singularities, as seen from eqns (1) and (3), are quoted as follows: (1) the r^{δ_r} singularity appears as $\delta_i = 0$ and $-1 < \delta_r < 0$; (2) the $(\ln r)^n$ singularity happens as $\delta_i \neq 0$ and $\delta_r = 0$; (3) the $r^{\delta_r}(\ln r)^n$ singularity ($n = 1, 2\ldots$) exists as $\delta_i \neq 0$ and $\delta_r > -1$.

Since the stress distribution in the interior region can be adequately described by the general solution for the corresponding $\delta = 0$ and the singular terms of the near-field solutions should have a weak effect on the stress field far away from the free edge, other non-singular terms of the general solutions ($\delta_i = 0$ and $\delta_r > 0$) are negligible in the analysis. In addition, although the $r^{\delta_r}(\ln r)^n$ stress singularity also exists as $\delta_i \neq 0$ and $\delta_r > 0$, the strength in this range is much smaller than other singularities. Hence, even though the transcendental characteristic eqns (8) and (9) have infinite roots (or eigenvalues) δ_n, only the ones with δ_r in the range of $-1 < \delta_r \leqslant 0$ are the main concern.

The complete stress and displacement field of the entire domain can be obtained by combining the stresses and displacements computed from all possible eigenvalues δ_n and corresponding C_k in conjunction with the remote boundary conditions far away from the free edge by numerical techniques. To comprehend the order and types of stress singularity near the free edge, however, the eigenvalues δ_n of the transcendental characteristic equation are worthy of thorough examination.

4. RESULTS AND DISCUSSIONS

To evaluate the free-edge stress singularity quantitatively, the commonly used graphite–epoxy laminates are taken as examples for examination. The elastic properties of each graphite–epoxy ply are shown as (Chen and Huang, 1993a, b):

$$E_{11} = E_{22} = 14.5 \quad \text{(GPa)}$$
$$E_{33} = 138 \quad \text{(GPa)}$$
$$G_{12} = G_{23} = G_{31} = 5.9 \quad \text{(GPa)}$$

and

$$\nu_{21} = \nu_{31} = \nu_{32} = 0.21,$$

where the subscripts 1, 2 and 3 refer to the transverse, thickness and fiber directions of an individual ply, respectively. E_{ii}, G_{ij} and ν_{ij} are Young's moduli, shear moduli and Poisson's ratios, respectively. For the graphite–epoxy laminates, the material constants μ_k, η_k, $q_{k\mu}$, ζ_k, ξ_k, $q_{k\zeta}$ and $t_{k\xi}$ are shown to be imaginary and $p_{k\mu}$, $t_{k\mu}$ and $p_{k\zeta}$ are real (Wang and Choi, 1982; Zwiers *et al.*, 1982).

4.1. *General* $[\theta_1/\theta_2]$ *graphite–epoxy laminates*

From the general solution as seen in eqn (1), it is seen that the asymptotic stress field near the corner of the free-edge interface is governed by the singular terms. The strength of the stress singularity δ is determined from the solutions of the transcendental characteristic equation as presented in eqn (8). As compared with the free-edge stress singularities for graphite–epoxy angle-ply laminates illustrated in Chen and Huang (1993b), the entirely different results for other commonly used laminates are also shown in Figs 3 and 4. Results reveal that graphite–epoxy angle-ply laminates have distinctive characteristics in the $[\theta_1/\theta_2]$ laminates. Since the material constants μ_k for the $(+\theta)$ ply are the same as those of the $(-\theta)$ ply, as seen in Chen and Huang (1993b), the characteristic determinant $|\mathbf{K}(\delta)|$ for angle-ply laminates is derived as a 6×6 determinant. From Figs 3 and 4, it is found that the r^{δ_r} and $(\ln r)^n$ singularities disappear occasionally in some laminates such as [30°/15°], [45°/15°].... Only the r^{δ_r} singularity goes out of sight in some laminates such as [60°/15°], [75°/45°].... The complete stress and displacement field of the entire domain can be obtained by using proper eigenvalues δ_n and corresponding C_k in conjunction with the remote boundary conditions far away from the free edge by numerical techniques.

4.2. $[\theta/0°]$ *and* $[\theta/90°]$ *graphite–epoxy laminates*

For the $[\theta/0°]$ and $[\theta/90°]$ laminates, the strength of stress singularity δ is determined from the transcendental characteristic equation as expressed in eqn (9). Except the free-edge stress singularity of graphite–epoxy cross-ply laminate presented in Chen and Huang (1993a), the results for the commonly used $[\theta/0°]$ and $[\theta/90°]$ graphite–epoxy laminates are shown in Fig. 5. Although there are three elementary stress singularities which were mentioned in Section 3, physically, the $(\ln r)^n$ singularity disappears for those cases. The other two elementary stress singularities, r^{δ_r} and $r^{\delta_r}(\ln r)^n$, will exist depending on the material elastic constants and fiber orientations of adjacent plies in the composite laminate.

From the results of the curves seen in Figs 3–5, it is evident that the free-edge stress singularities for any composite laminates is expected to obtain a continuous spectrum. Although the strength of those stress singularities is hard to compare, the numerical experiment shows that the asymptotic stress field is mainly controlled by the coefficients of the singular terms which are determined by the aid of the remote boundary conditions far

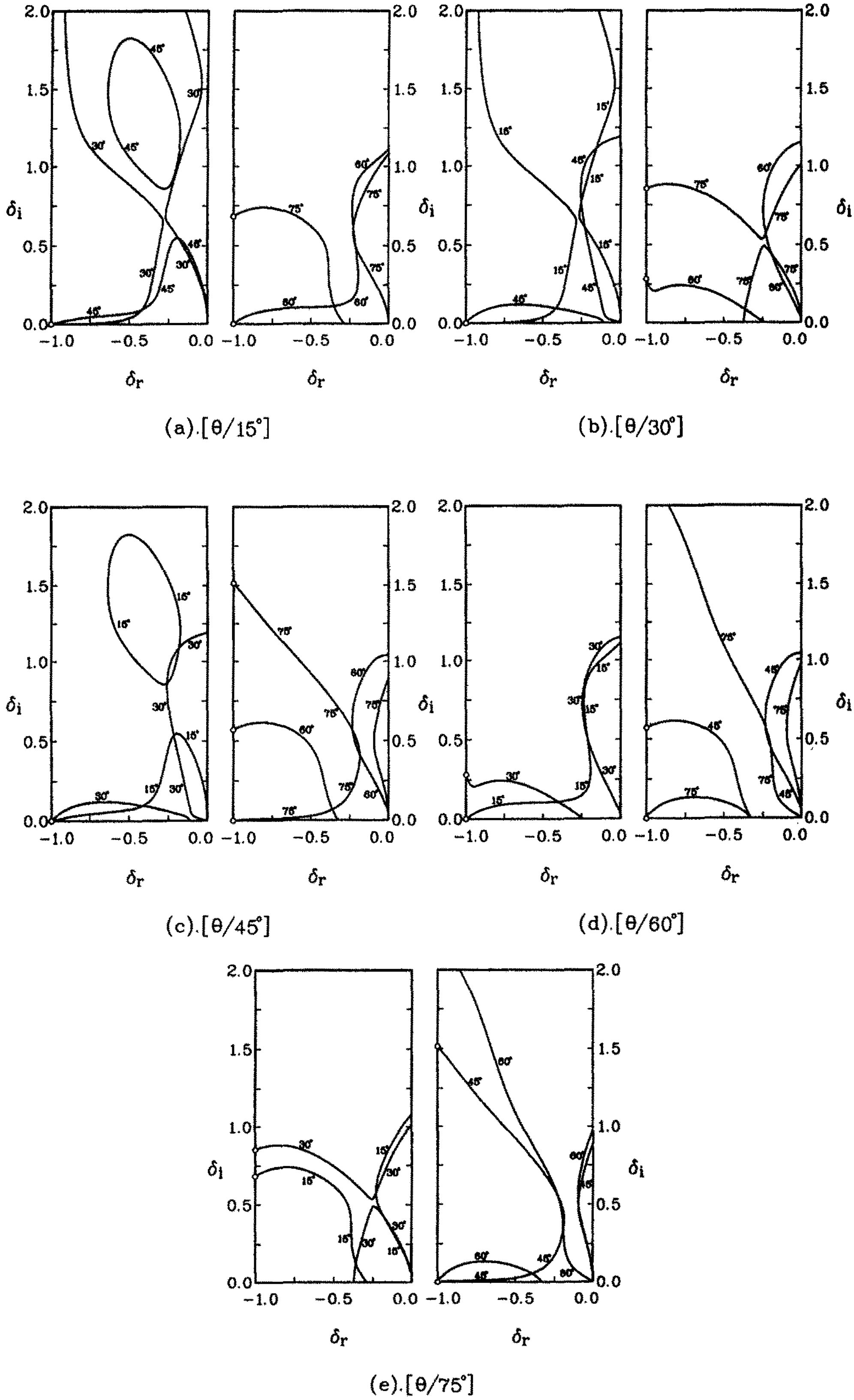

Fig. 3. Dominant free-edge stress singularity of [θ/15°], [θ/30°], [θ/45°], [θ/60°] and [θ/75°] graphite–epoxy laminates.

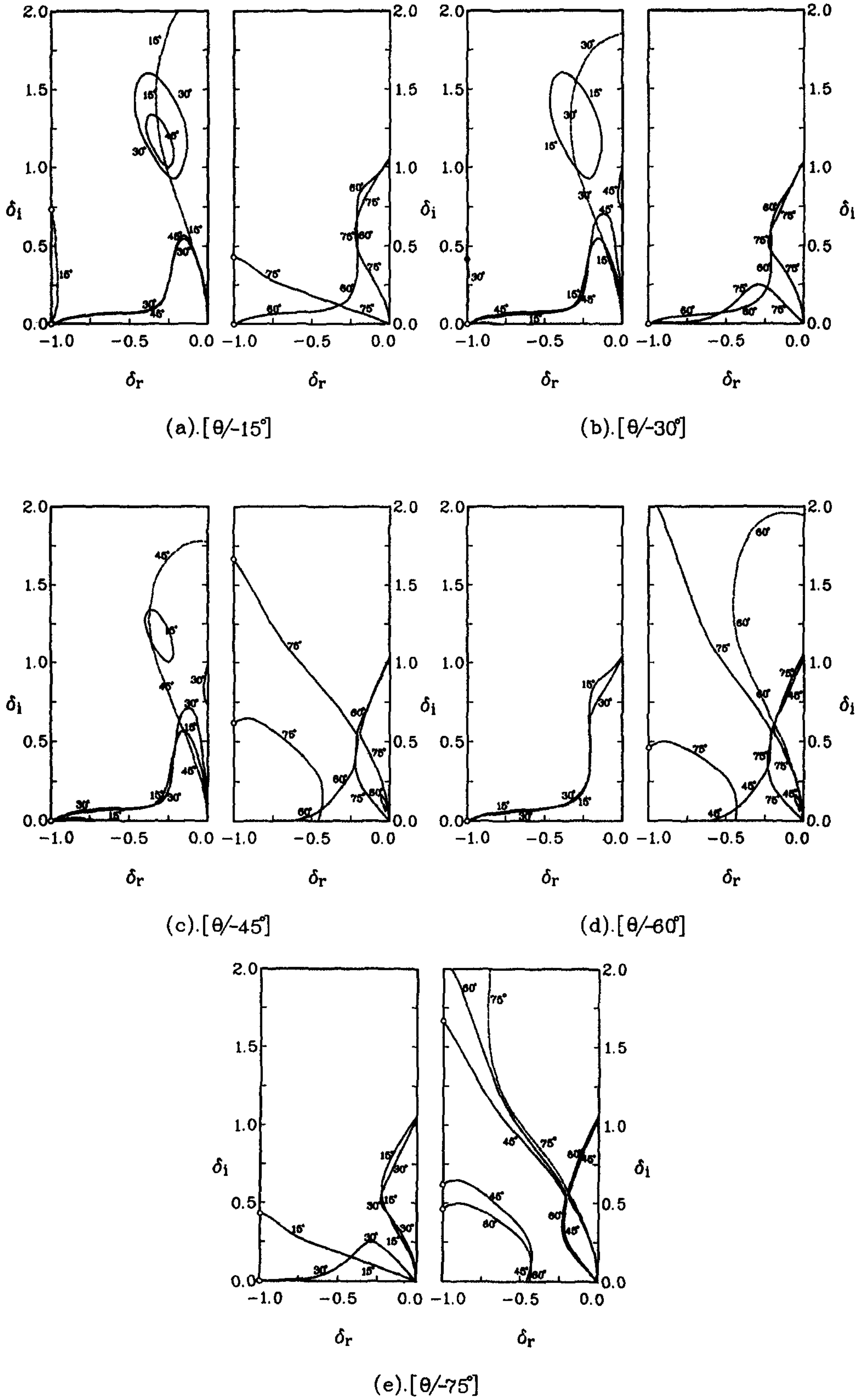

Fig. 4. Dominant free-edge stress singularity of $[\theta/-15°]$, $[\theta/-30°]$, $[\theta/-45°]$, $[\theta/-60°]$ and $[\theta/-75°]$ graphite–epoxy laminates.

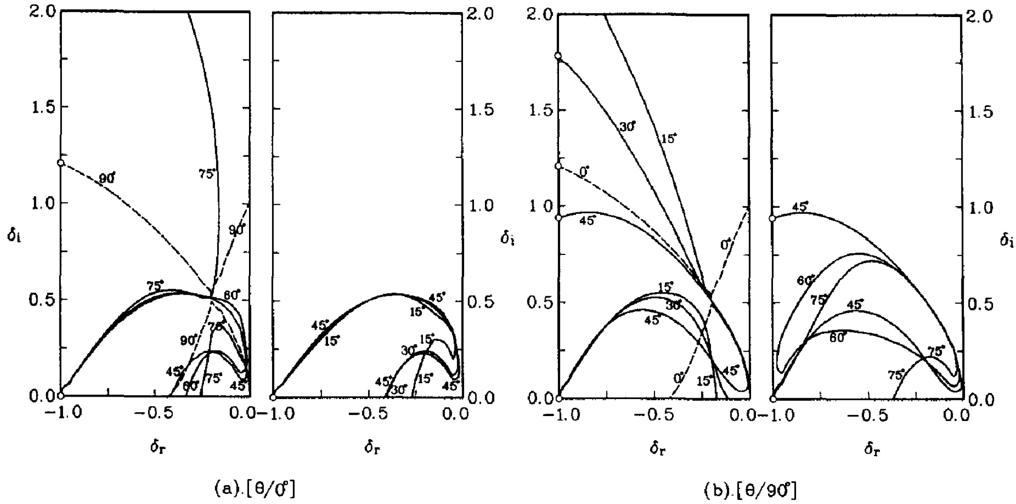

Fig. 5. Dominant free-edge stress singularity of [θ/0°] and [θ/90°] graphite–epoxy laminates.

away from the free edge. These results will assist the design against the failure of composite materials and structures.

5. CONCLUSIONS

A rigorous investigation of the free-edge stress singularity of the general composite laminates under uniform axial strain has been presented. Once the ply material properties and stacking sequence are known, the accurate characteristics of the free-edge stress singularity of examined composite laminates can be obtained. The cases tackled include the commonly used [θ_1/θ_2], [θ/0°] and [θ/90°] graphite–epoxy laminates. The results can be used to illustrate the fundamental nature of the free-edge effects in related composite materials. The present developed approach can be further extended to deal with the delamination-crack and transverse-crack of composite laminates and will be presented in a subsequent report.

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REFERENCES

Bogy, D. B. (1968). Edge-bonded dissimilar othogonal elastic wedges under normal and shear loading. *J. Appl. Mech.* **35,** 460–466.
Chen, W. H. and Huang, T. F. (1993a). Analysis of free-edge effects in cross-ply laminates under uniform axial extension. *J. Appl. Mech.* (submitted).
Chen, W. H. and Huang, T. F. (1993b). On the free-edge effects in angle-ply laminates under uniform axial strain. *Int. J. Solids Structures* (submitted).
Hein, V. L. and Erdogan, F. (1971). Stress singularities in a two-material wedge. *Int. J. Fract. Mech.* **7,** 317–330.
Kuo, M. C. and Bogy, D. B. (1974). Plane solutions for traction problems on orthotropic unsymmetrical wedges and symmetrically twinned wedges. *J. Appl. Mech.* **41,** 203–208.
Lekhnitskii, S. G. (1963). *Theory of Elasticity of an Anisotropic Body.* Holden-Day, San Francisco.
Muller, D. E. (1956). A method for solving algebraic equations using an automatic computer. *Mathl. Tabl. Comput.* **10,** 208–215.
Pavelle, R. and Wang, P. S. (1985). Macsyma from f to g. *J. Symbolic Comput.* **1,** 69–100.
Reissner, E. (1944). Note on the theorem of the symmetry of the stress tensor. *J. Math. Phys.* **23,** 192–194.
Stroh, A. N. (1962). Steady state problems in anisotropic elasticity. *J. Math. Phys.* **41,** 77–103.
Wang, S. S. and Choi, I. (1982). Boundary-layer effects in composite laminates. Part 1 : free-edge stress singularities. *J. Appl. Mech.* **49,** 541–548.
Yin, W. L. (1991). Free-edge effects in laminates under extension, bending and twisting. Part 1 : a stress function approach. *Proc. 32nd Structures, Structural Dynamics and Materials Conference,* pp. 985–995. Baltimore, MD.
Zwiers, R. I., Ting, T. C. T. and Spilker, R. L. (1982). On the logarithmic singularity of free-edge stress in laminated composites under uniform extension. *J. Appl. Mech.* **49,** 561–569.

APPENDIX

The determinant $|\mathbf{K}_d(\delta)|$ *for the* $[\theta/0°]$ *and* $[\theta/90°]$ *laminates*

$$
|\mathbf{K}_d(\delta)| = \begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\mu_{1(m)} & \bar{\mu}_{1(m)} & \mu_{2(m)} & \bar{\mu}_{2(m)} & \mu_{3(m)} & \bar{\mu}_{3(m)} & \zeta_{1(m+1)} & \bar{\zeta}_{1(m+1)} & \zeta_{2(m+1)} & \bar{\zeta}_{2(m+1)} \\
p_{1\mu}^{(m)} & \bar{p}_{1\mu}^{(m)} & p_{2\mu}^{(m)} & \bar{p}_{2\mu}^{(m)} & p_{3\mu}^{(m)} & \bar{p}_{3\mu}^{(m)} & p_{1\zeta}^{(m+1)} & \bar{p}_{1\zeta}^{(m+1)} & p_{2\zeta}^{(m+1)} & \bar{p}_{2\zeta}^{(m+1)} \\
q_{1\mu}^{(m)} & \bar{q}_{1\mu}^{(m)} & q_{2\mu}^{(m)} & \bar{q}_{2\mu}^{(m)} & q_{3\mu}^{(m)} & \bar{q}_{3\mu}^{(m)} & q_{1\zeta}^{(m+1)} & \bar{q}_{1\zeta}^{(m+1)} & q_{2\zeta}^{(m+1)} & \bar{q}_{2\zeta}^{(m+1)} \\
A_{11}^{(m)} & \bar{A}_{11}^{(m)} & A_{12}^{(m)} & \bar{A}_{12}^{(m)} & A_{13}^{(m)} & \bar{A}_{13}^{(m)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \zeta_{1(m+1)}^{\delta+2} & \bar{\zeta}_{1(m+1)}^{\delta+2} & \zeta_{2(m+1)}^{\delta+2} & \bar{\zeta}_{2(m+1)}^{\delta+2} \\
A_{21}^{(m)} & \bar{A}_{21}^{(m)} & A_{22}^{(m)} & \bar{A}_{22}^{(m)} & A_{23}^{(m)} & \bar{A}_{23}^{(m)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \zeta_{1(m+1)}^{\delta+1} & \bar{\zeta}_{1(m+1)}^{\delta+1} & \zeta_{2(m+1)}^{\delta+1} & \bar{\zeta}_{2(m+1)}^{\delta+1} \\
A_{31}^{(m)} & \bar{A}_{31}^{(m)} & A_{32}^{(m)} & \bar{A}_{32}^{(m)} & A_{33}^{(m)} & \bar{A}_{33}^{(m)} & 0 & 0 & 0 & 0 \\
H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & 0 & 0 & 0 & 0
\end{vmatrix}
$$

$$
= \sum T_{ijklmn} B_j B_k B_l A_m, A_n,
$$

where

$$
\begin{aligned}
T_{ijklmn} = (-1)^{i+j+k+l+m+n} & H_i U_j U_k U_l K_m K_n (K_m - K_n) * \operatorname{sign}(j-i) * \operatorname{sign}(k-i) \\
& * \operatorname{sign}(l-i) * [U_j(V_k - V_l) + U_k(V_l - V_j) + U_l(V_j - V_k)] \\
& * \{U_{n1}[(Q_{n2} - S_{n3})(R_{n3} - R_{n4}) - (P_{n2} - R_{n3})(S_{n3} - S_{n4})] \\
& - U_{n2}[(S_{n3} - S_{n4})(R_{n4} - P_{n1}) - (R_{n3} - R_{n4})(S_{n4} - Q_{n1})] \\
& + K_{n3}[(S_{n4} - Q_{n1})(P_{n1} - P_{n2}) - (R_{n4} - P_{n1})(Q_{n1} - Q_{n2})] \\
& - K_{n4}[(Q_{n1} - Q_{n2})(P_{n2} - R_{n3}) - (P_{n1} - P_{n2})(Q_{n2} - S_{n3})]\} \\
& 1 \leqslant i, j, k, l, n1, n2 \leqslant 6, \quad i \neq j \neq k \neq l \neq n1 \neq n2, \quad l > k > j, \quad n2 > n1 \\
& 1 \leqslant m, n, n3, n4 \leqslant 4, \quad m \neq n \neq n3 \neq n4, \quad n > m, \quad n4 > n3
\end{aligned}
$$

$$
H_i = \frac{[W_{i(m)}(\bar{\zeta}_{1(m+1)}^{\delta+1} - \zeta_{1(m+1)}^{\delta+1}) + V_{i(m)}(\bar{t}_{1\zeta}^{(m+1)} \zeta_{1(m+1)}^{\delta+1} - t_{1\zeta}^{(m+1)} \bar{\zeta}_{1(m+1)}^{\delta+1})]}{(t_{1\zeta}^{(m+1)} - \bar{t}_{1\zeta}^{(m+1)})}
$$

$$
\begin{aligned}
& B_1 = \mu_{1(m)}^{\delta}, \quad B_2 = \bar{\mu}_{1(m)}^{\delta}, \quad B_3 = \mu_{2(m)}^{\delta}, \quad B_4 = \bar{\mu}_{2(m)}^{\delta}, \quad B_5 = \mu_{3(m)}^{\delta}, \quad B_6 = \bar{\mu}_{3(m)}^{\delta} \\
& U_1 = \mu_{1(m)}, \quad U_2 = \bar{\mu}_{1(m)}, \quad U_3 = \mu_{2(m)}, \quad U_4 = \bar{\mu}_{2(m)}, \quad U_5 = \mu_{3(m)}, \quad U_6 = \bar{\mu}_{3(m)} \\
& V_1 = \eta_{1(m)}, \quad V_2 = \bar{\eta}_{1(m)}, \quad V_3 = \eta_{2(m)}, \quad V_4 = \bar{\eta}_{2(m)}, \quad V_5 = \eta_{3(m)}, \quad V_6 = \bar{\eta}_{3(m)} \\
& W_1 = t_{1\mu}^{(m)}, \quad W_2 = \bar{t}_{1\mu}^{(m)}, \quad W_3 = t_{2\mu}^{(m)}, \quad W_4 = \bar{t}_{2\mu}^{(m)}, \quad W_5 = t_{3\mu}^{(m)}, \quad W_6 = \bar{t}_{3\mu}^{(m)} \\
& P_1 = p_{1\mu}^{(m)}, \quad P_2 = \bar{p}_{1\mu}^{(m)}, \quad P_3 = p_{2\mu}^{(m)}, \quad P_4 = \bar{p}_{2\mu}^{(m)}, \quad P_5 = p_{3\mu}^{(m)}, \quad P_6 = \bar{p}_{3\mu}^{(m)} \\
& Q_1 = q_{1\mu}^{(m)}, \quad Q_2 = \bar{q}_{1\mu}^{(m)}, \quad Q_3 = q_{2\mu}^{(m)}, \quad Q_4 = \bar{q}_{2\mu}^{(m)}, \quad Q_5 = q_{3\mu}^{(m)}, \quad Q_6 = \bar{q}_{3\mu}^{(m)}
\end{aligned}
$$

$$
\begin{aligned}
& A_1 = \zeta_{1(m+1)}^{\delta}, \quad A_2 = \bar{\zeta}_{1(m+1)}^{\delta}, \quad A_3 = \zeta_{2(m+1)}^{\delta}, \quad A_4 = \bar{\zeta}_{2(m+1)}^{\delta} \\
& K_1 = \zeta_{1(m+1)}, \quad K_2 = \bar{\zeta}_{1(m+1)}, \quad K_3 = \zeta_{2(m+1)}, \quad K_4 = \bar{\zeta}_{2(m+1)} \\
& R_1 = p_{1\zeta}^{(m+1)}, \quad R_2 = \bar{p}_{1\zeta}^{(m+1)}, \quad R_3 = p_{2\zeta}^{(m+1)}, \quad R_4 = \bar{p}_{2\zeta}^{(m+1)} \\
& S_1 = q_{1\zeta}^{(m+1)}, \quad S_2 = \bar{q}_{1\zeta}^{(m+1)}, \quad S_3 = q_{2\zeta}^{(m+1)}, \quad S_4 = \bar{q}_{2\zeta}^{(m+1)}.
\end{aligned}
$$